Dynamics of Eulerian walkers

A. M. Povolotsky, V. B. Priezzhev, and R. R. Shcherbakov
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia
(Received 19 December 1997)

We investigate the dynamics of Eulerian walkers as a model of self-organized criticality. The evolution of the system is divided into characteristic periods which can be seen as avalanches. The structure of avalanches is described and the critical exponent in the distribution of first avalanches $\tau=2$ is determined. We also study a mean square displacement of Eulerian walkers and obtain a simple diffusion law in the critical state. The evolution of an underlying medium from a random state to the critical one is described.
[S1063-651X(98)14310-6]

PACS number(s): 05.70.Ln, 05.40.+j, 02.70.−c

I. INTRODUCTION

To illustrate the phenomenon of self-organized criticality (SOC) [1] a wide range of cellular automata such as sand piles, rice piles, and forest fires have been proposed [1–3]. They assume a system consisting of a large number of elements. The energy being randomly added to the system is redistributed then over the degrees of freedom by a kind of nonlinear diffusion. This is realized by avalanchelike processes which carry the added energy out of the system. As a rule, the system spontaneously evolves towards the critical state free of any characteristic length and time scale. In this state, probabilistic distributions of quantities characterizing the statistical ensemble exhibit the power law behavior.

Which features of the SOC dynamics are responsible for the existence of a dynamic attractor in complex systems? What are the origins of the scaling and self-similarity in the stationary state? To answer these questions, one should investigate nonlinear diffusion in the SOC models and study the structure of avalanches. The difficulties encountered here arise from the complexity of dynamic processes in the strongly correlated SOC systems. Up to now, the most analytically tractable model has been the Abelian sand pile model (ASM) [4]. Due to its simple algebraic structure, the detailed description of the SOC state of ASM has been given, and some critical exponents have been found [5–8]. Recently, a new model has been proposed which is called the Eulerian walkers model (EWM) [9]. In a sense, this model is even more elementary than ASM as it deals with a single moving particle. The dynamics of this model is driven by a walking particle. The motion of a particle is affected by the medium, and in its turn affects the medium inducing long range correlations in the system. If the walk occurs in a closed system, it continues infinitely long and eventually gets self-organized into Eulerian trails [10]. If a system is open, the particles can leave the system and new particles drop time after time. In this case, the system evolves to the critical state similar to that in ASM. By analogy with ASM, the avalanches in EWM have been introduced [11] as periods of reconstruction of recurrent states, after they have been broken by an added particle.

Another aspect of EWM is the possibility to look at nontrivial diffusion laws and their change under the self-organization. In contrast to the self-avoiding walk where an infinitely long memory is due to exclusion of multiple visits of lattice sites, EWM presents an alternative way to introduce memory effects. The visited sites are not forbidden for the next visits but a prescription for the next step is changed after each visit. As a result, EWM evolves to the critical state where the deterministic walk is characterized by the simple diffusion law.

Like most of the problems of the graph theory, EWM admits a simple "real life" interpretation. A treatment of EWM as the model of the distribution of goods in a spatially extended market is given in Sec. II.

The article is organized as follows. In Sec. II the algebraic structure of the critical state of EWM is described by using the analogy with ASM. In Sec. III the definition of avalanches in EWM is given. The structure of avalanches is described and the critical exponent of the distribution of first avalanche sizes is obtained. In Sec. IV the evolution of the system as a whole in recurrent and transient states is discussed and the mean square displacement of the particle with time is described for both of them. Analytic results are supported by Monte Carlo simulations.

II. ALGEBRAIC PROPERTIES OF EULERIAN WALKERS MODEL

The Eulerian walkers model is defined as follows. Consider an arbitrary connected graph **G** consisting of N sites. Each site of **G** is associated with an arrow which is directed along one of the incident bonds. The arrow directions at the site i are specified by the integers α_i $(1\leqslant\alpha_i\leqslant\tau_i)$ where τ_i is the number of nearest neighbors of the site i. The set $\{\alpha_i\}$ gives a complete description of the medium. Starting with an arbitrary arrow configuration one drops the particle to a site of **G** chosen at random. At each time step (i) the particle arriving at a site i changes the arrow direction from α_i to α_i+1, if $\alpha_i<\tau_i$ and to 1, if $\alpha_i=\tau_i$; and (ii) the particle moves one step along the new arrow direction from i to the neighboring site i'.

Having no end points on **G**, the particle continues to walk infinitely long. Due to a finite number of possible states of the system, it eventually settles into the Poincaré cycle. For most dynamic systems the recurrence time of this cycle grows exponentially with N. It has been shown in [9] that for the EWM the Poincaré cycle is squeezed to the Eulerian trail

[10] with the recurrence time of an order of N. During the Eulerian trail the particle passes all bonds of the graph exactly once in each direction.

There is the following simple interpretation of the dynamic rules of the model. Let us consider a traveling merchant who buys and sells different kinds of goods in towns connected by roads. All towns are supposed to produce different kinds of goods. Upon arrival at a town, he sells all the goods bought in the previous town and buys the new ones to be sold in a next town. Having bought some goods in a town, say Dubna, the merchant goes to one of its nearest neighbors. Actually, this town is not random. The merchant has a simple strategy to ensure the highest level of sales revenues. First of all, if he never sold the goods produced in Dubna in some of the neighboring towns, he chooses one of them (arbitrary) to visit next. Then, if all the neighbors have already been visited, he prefers that neighbor where he last sold the goods produced in Dubna earlier than at all the other neighbors. If towns and roads connecting them are considered as sites and bonds of a graph, respectively, then the merchant motion matches the rules of EWM dynamics.

Let **G** be an open graph. It means that one auxiliary site is introduced called a sink. The subset of sites of **G** connected with the sink forms an open boundary. The sink has no arrow and the particle reaching the sink leaves the system. Then, the new particle is dropped to a site of **G** chosen at random. Since on the closed graph the particle visits all sites during the walk, at the open graph it always reaches the sink. A set $\{C\}$ of configurations $C=\{\alpha_j\}$ which remains on **G** when the particle left **G** for the sink is the set of stable configurations. The operator a_i can be introduced as follows:

$$a_i C = C', \tag{1}$$

which describes the resulting transformation caused by dropping the particle to the site i. As usual in the theory of Markov chains, the set $\{C\}$ may be divided into two subsets. The first subset denoted by $\{R\}$ includes those configurations which can be obtained from an arbitrary configuration by a sequential action of the operators a_i. It follows from the definition that the subset $\{R\}$ is closed under a multiple action of the operators a_i. Once the system gets into $\{R\}$, it never gets out under subsequent evolution. All nonrecurrent configurations are called transient and form the subset $\{T\}$ which is the complement to the set $\{R\}$. By definition, any recurrent configuration $C\in\{R\}$ may be reached from another $C'\in\{R\}$ by a subsequent action of the operators a_i. Since this is valid for $C'=C$ too, the identity operator acting in $\{R\}$ exists. In addition, the operators a_i have the following properties.

(1) For arbitrary sites i and j and for any configuration of arrows C

$$a_i a_j C = a_j a_i C. \tag{2}$$

(2) For any recurrent configuration $C\in\{R\}$, there exists a unique operator

$$(a_i^{-1}C)\in\{R\}$$

such that

$$a_i(a_i^{-1})C = C. \tag{3}$$

The proof of these statements is similar to the one for the avalanche operators in ASM [4] and is given in [6]. Thus the operators a_i acting in the set of recurrent configurations $\{R\}$ form the Abelian group. The addition of τ_i particles to site i gives the same effect as the addition of one particle to each of τ_i neighbors of i. It returns the arrow outgoing from i to the former position and initiates the motion of one particle to each neighboring site. In the operator form, we have

$$a_i^{\tau_i} = \prod_{k=1}^{\tau_i} a_{j_k}, \tag{4}$$

where j_k are neighbors of the site i. Introducing the discrete Laplacian on **G** as

$$\Delta_{ij} = \begin{cases} \tau_i, & i=j \\ -1, & i \text{ and } j \text{ are connected by a bond} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and using Eq. (4), one can write the identity operator as

$$E_i = \prod_{j\in\mathbf{G}} a_j^{\Delta_{ij}}. \tag{6}$$

Since all recurrent configurations can be obtained from an arbitrary one by a successive action of operators a_i, one can represent any $C\in\{R\}$ in the form

$$C = \prod_{i\in\mathbf{G}} (a_i)^{n_i} C^*. \tag{7}$$

The N-dimensional vector $\mathbf{n}$ labels all possible recurrent configurations. Equation (6) shows that two vectors $\mathbf{n}$ and $\mathbf{n}'$ label the same configuration if the difference between them is $\Sigma_j m_j \Delta_{ij}$ where m_j are integers. The N-dimensional space $\{\mathbf{n}\}$ has a periodic structure with an elementary cell of the form of a hyper-parallelepiped with base edges $\vec{e}_i = (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN})$. Thus, the number of nonequivalent recurrent configurations is

$$N = \det\Delta, \tag{8}$$

which is the Kirchhoff formula [10] for spanning trees and the Dhar formula for the number of recurrent configurations in ASM [4]. The correspondence to ASM is not surprising. The algebra of the operators a_i completely coincides with that of avalanche operators of the Abelian sand pile model [4]. Moreover, the identity operator (6) has the same form for both the models.

Continuing the analogy between recurrent configurations of EWM and sand piles, one can find the expected number G_{ij} of full rotations of the arrow at site j, due to the particle dropped at i [4]. During the walk, the expected number of steps from j is $\Delta_{jj}G_{ij}$ whereas $-\Sigma_{k\neq j} G_{ik}\Delta_{kj}$ is the average flux into j. Equating both the fluxes, one gets

$$\sum_k G_{ik}\Delta_{kj} = \delta_{ij} \tag{9}$$

or

$$G_{ij}=[\Delta^{-1}]_{ij}. \tag{10}$$

The expected number of full rotations of the arrow is equal to the number of entries into the site j divided by τ_j. On the other hand, the number of visits of a site for the random walk is also the Green function of the Laplace equation. Thus we have a surprising fact that the number of visits of the site by the particle for the deterministic motion in EWM coincides with that for the ordinary random walk.

The direct correspondence between spanning trees, recurrent configurations of EWM, and Eulerian trails can be established in the following way. The walking particle leaves each site along an arrow after turning this arrow. Therefore the trajectory of the particle is traced by arrows. If **G** is an open graph, all trajectories end at the sink and never form loops producing acyclic configurations of arrows.

Given an acyclic arrow configuration, we can construct a unique spanning tree rooted in the sink and vice versa. Indeed, all bonds along which the arrows are directed form the spanning tree. Conversely, if we have a spanning tree rooted in the sink we can obtain the acyclic arrow configuration by pointing the arrow from each site along the path leading to the sink. This correspondence allows us to identify the acyclic arrow configurations and spanning trees. Below, we do not distinguish between them and by a spanning tree we mean both the spanning tree and its arrow representation.

If **G** is the closed graph, the particle settles into the Eulerian trail during which it passes each bond exactly once in each direction. Let the particle which has already visited all sites arrive at a site i at some moment. If we now remove the arrow from i, we obtain the acyclic arrow configuration where all arrow paths lead to i. This defines the spanning tree rooted in the site of the current particle location.

III. AVALANCHE DYNAMICS

The particle added to the recurrent configuration of ASM may induce successive topplings of sites called the avalanche. At the initial moment, it destroys the recurrent configuration and the system leaves the recurrent state. After the avalanche stops, the recurrent configuration is restored again. Thus the avalanche in ASM can be defined as a period of reconstruction of the recurrent state. This definition may be directly applied to EWM.

We start with a recurrent state of EWM. The corresponding arrow configuration forms a spanning tree. Once a particle is dropped, it can destroy the spanning tree by closing a loop of arrows. During the evolution, one loop can be transformed into another. When all loops disappear, the spanning tree is restored. The interval of existence of the loop can be called the *avalanche of cyclicity* or simply avalanche. The loops can be created and destroyed several times during the motion of one particle. Therefore, unlike ASM, an addition of one particle can initiate several avalanches in the system. When a particle comes to the sink, it always directs the arrows to the sink thus restoring the spanning tree. Therefore, when the particle leaves the system, the avalanche always ends and the recurrent state is restored. The particle dynamics represents successive transitions from one recurrent state to another through avalanches. To study the evolution of the

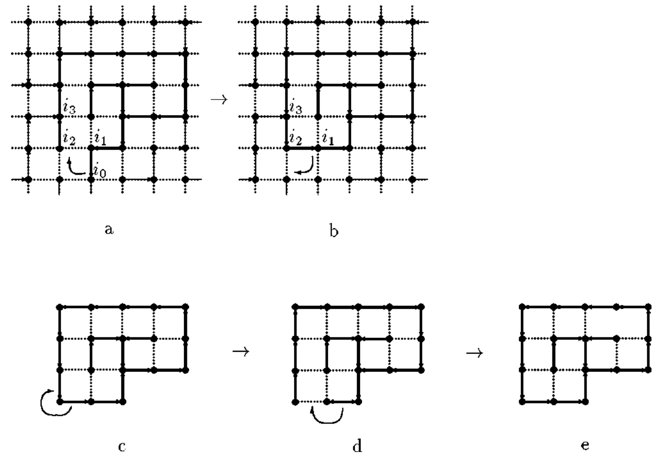

FIG. 1. (a) Closing the loop at i_1. (b) The last step before openning the loop. (c), (d), (e) The evolution on the closed graph settled into the Eulerian trail. The loops in (a) and (b) exactly coincide with those in (d) and (e).

system, the structure of the avalanche should be considered in detail.

Consider the Eulerian walk on the square lattice $\mathcal{L}$ of size $L\times L$ with open boundary conditions. Each boundary site is connected to the sink by one bond on the edge and by two bonds at the corners of $\mathcal{L}$. The rule of arrow rotations is the same for all sites. If we denote the bonds outgoing from a site i by N,E,S,W, the rule of rotations is $N\rightarrow E\rightarrow S\rightarrow W$. In other words, when the particle arrives at a site, the arrow outgoing from this site turns to the next bond clockwise. For a topological reason, this rule leads to a simple structure of avalanches, namely, to clusters of sites visited by a particle, being compact.

Let the particle be dropped to a recurrent configuration which is a spanning tree. At some step the first loop is created. The arrows can form loops of two kinds: clockwise and anticlockwise. The loop is clockwise if tracing the loop along arrows leaves the interior of the loop on the right and anticlockwise otherwise. It is easy to see that due to the clockwise rule of rotations, only clockwise loops can be created from recurrent states. Indeed, the anti-clockwise loop arises when the arrow, which closes this loop, is directed at the previous time step into the area bounded by the loop. The arrow path beginning from this arrow could not leave the area of the loop without intersections with the loop. This means that before this loop was closed, another loop existed, which contradicts the assumption that we start with a spanning tree.

Consider the evolution after closing a clockwise loop at the spanning tree. Denote by ij the arrow if it is pointed from site i to site j. Analogously, we denote by $i_1i_2i_3\ldots$ the arrow path if the arrow from site i_1 is pointed to site i_2, the arrow from i_2 is pointed to i_3, and so on. Let a spanning tree exist at the time step $(t-1)$, while at the step t, the particle that arrived at the site i_1 changes the arrow direction from i_1i_0 to i_1i_2 and the clockwise loop $\mathcal{O}^+=i_1i_2i_3\ldots i_ni_1$ appears [Fig. 1(a)]. Now, we can prove the following proposition.

Proposition 1. The particle does not leave the area of the loop $\mathcal{O}^+$ and the spanning tree cannot be restored until all arrows inside the loop area make the full rotation and the arrows belonging to the loop itself change the direction to anticlockwise forming the anticlockwise loop $\mathcal{O}^{-} = i_1 i_n \ldots i_2 i_1$. At the last step when $\mathcal{O}^{-}$ appears, the particle arrives at i_2 and at the next step the arrow at i_2 rotates out of the loop area and the spanning tree may be restored [Fig. 1(b)].

Proof. Consider EWM on the auxiliary graph $\mathcal{G}$, which is a part of the square lattice bounded by the loop $\mathcal{O}^{-}$ with closed boundary conditions. The closed boundary means that all bonds that link boundary sites $i_1, i_2, i_3, \ldots, i_n$ with the sites of the lattice outside the loop area are removed. The rules of rotations are modified so that an arrow skips deleted bonds. We consider the Eulerian trail at $\mathcal{G}$ starting from the site i_2. At the initial moment, the arrow configuration at $\mathcal{G}$ differs from that on the lattice $\mathcal{L}$ only by orientation of the loop: instead of the clockwise loop $\mathcal{O}^{+} = i_1 i_2 i_3 \ldots i_n i_1$ on $\mathcal{L}$, we have the anticlockwise loop $\mathcal{O}^{-} = i_1 i_n \ldots i_2 i_1$ on $\mathcal{G}$ [Fig. 1(c)]. Starting from the first step, $(n-1)$ successive steps reverse $\mathcal{O}^{-}$ into $\mathcal{O}^{+}$ and the particle arrives at i_1 [Fig. 1(d)]. Notice that the initial arrow configuration on $\mathcal{G}$ corresponds to that described in the preceding section, when the particle has already settled into the Eulerian trail on the closed graph. Indeed, at the first moment, all arrows except the arrow at the site of the current particle location form the spanning tree rooted of this site. Hence, the subsequent evolution leads again to the loop $\mathcal{O}^{-}$ via full rotation of arrows at all internal sites [Fig. 1(e)]. On the other hand, this part of evolution of the graph $\mathcal{G}$ coincides with the one on the original lattice $\mathcal{L}$ since the moment when the loop $\mathcal{O}^{+}$ is closed [Fig. 1(a)] up to the moment when it is changed by $\mathcal{O}^{-}$ [Fig. 1(b)]. At the last step, the arrow $i_2 i_1$ rotates out of the loop area and the loop can be broken. Before this moment the loop exists permanently as during the Eulerian trail one loop always exists. The proposition is proven.

Generally, the avalanche does not necessarily end after that. Two situations are possible. At the last step, the arrow at i_2 turns outside the anticlockwise loop $i_2 i_1 \rightarrow i_2 i_2'$. If i_1' is connected to the sink through the arrow path, the spanning tree is restored and the avalanche is finished. This is the case of a one-loop avalanche. In the other case, the arrow path from i_2' goes to i_2, i.e., i_2' is the predecessor of i_2 with respect to the sink. Then, one more loop is closed and the avalanche continues. This is a two-loop avalanche. The second loop relaxes like the first one. When the second loop is reversed, the spanning tree is always restored because at the last step the particle arrives at i_0 which was connected to the sink by an arrow path before the avalanche started.

Several consequences can be obtained from the picture described. During the avalanche the particle visits sites inside the loop four times, sites of the edge twice, and sites at the corners with angles of $\pi/2$ and $3\pi/2$ once and three times, respectively. Generally, if two arrows belonging to a loop, one of which comes to the site on the loop and the other goes out from this site, form the angle α, then the particle visits this site during an avalanche $2\alpha/\pi$ times. The sum of angles of corners of any loop on the square lattice is equal to $(\gamma-2)\pi$, where γ is the number of corners. Then, the number of steps necessary to cover a loop is given by the formula

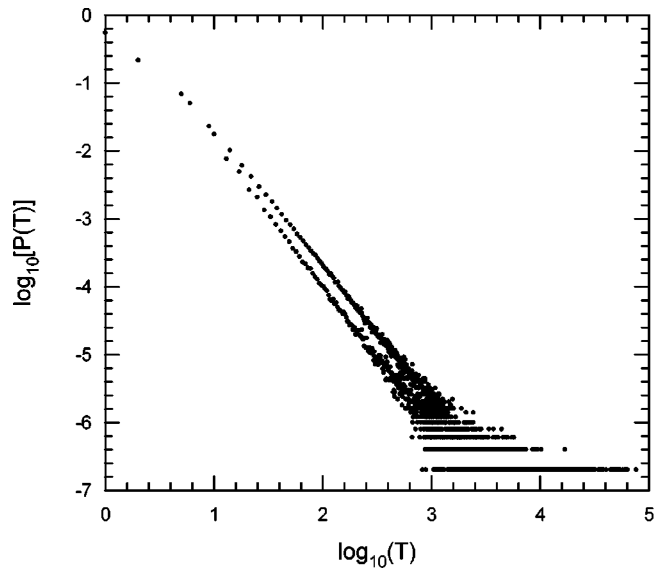

FIG. 2. The distribution of duration of the first avalanche in the SOC state is shown on the double logarithmic plot. The distribution splits into two parts as described in the text. The slope of both the parts is the same with the critical exponent $\tau = 2.0$.

$$T = [4s + 2(p-\gamma) + 2(\gamma-2)] + 1 = (4s + 2p - 4) + 1, \tag{11}$$

where s is the number of inner sites, and p is the perimeter of the loop. As the avalanches can consist of one or two loops and the perimeter p is always even, the duration of avalanches can be equal to any of the following numbers:

$$\begin{aligned} T_1 &= 1(\mathrm{mod}\,4), \\ T_2 &= 2(\mathrm{mod}\,4), \end{aligned} \tag{12}$$

where T_1 and T_2 are the durations of avalanches consisting of one and two loops, respectively. This explains the double distribution of durations of avalanches (Fig. 2) obtained in [12]. Also we can find the critical exponent of the duration distribution for the first avalanche. In the thermodynamic limit, the duration of avalanches grows as the area of the loop. It has been shown in [13] that the probability to get a loop of the size s when a bond is added to the spanning tree at random is equal to

$$P(s) \sim s^{-11/8}. \tag{13}$$

In the distribution (13), the loop is assumed to be linked to the sink by a unique path attached to an arbitrary site at its perimeter, wherever it is closed by the added bond. In our case, loops are closed by turning the arrow that was connected to the sink through an arrow path before the turn. Hence, for the loop of perimeter p, one should select only the latter case from p possible positions of the site linked to the sink with respect to the place where the loop is closed. To this end, the distribution (13) should be divided by the perimeter of the loop. Taking into account that the perimeter p of the loop scales with the linear size r as the fractal dimension of a chemical path on a spanning tree [13] $p \sim r^{5/4}$ and that the loop is compact $s \sim r^2$, we obtain

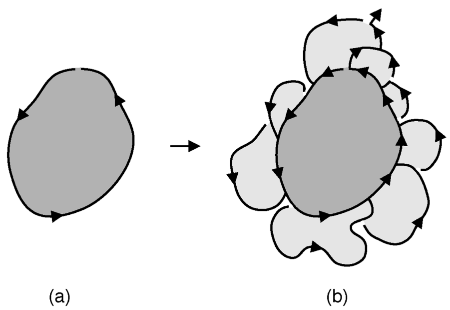

FIG. 3. A subsequent evolution of a cluster of visited sites in the SOC state. A schematic picture of the cluster after the first (a) and second (b) stages of evolution. The areas with different numbers of visits are shown by different colors. The directions of arrows correspond to their final positions.

$$\mathcal{P}(s) \sim \frac{s^{-11/8}}{r^{5/4}} \sim s^{-2}. \tag{14}$$

Thus for the first avalanches the critical exponent of the distribution of duration is $\tau = 2$. The one- and two-loop avalanches differ only in a local structure of the spanning tree at the site of closing the loop. Therefore the critical exponents are the same for both the distributions. This result is in excellent agreement with numerical simulations presented in Fig. 2 where we have considered the EWM on the square lattice of linear size $L = 400$ with open boundary conditions.

The first avalanches in EWM are similar to the erased loops in the loop-erased walks, studied in [14]. The same exponent $\tau = 2$ was obtained.

The result $\tau = 2$ is valid only for the first avalanches for their independence of each other. The analytic derivation of τ for arbitrary avalanches is a more difficult problem due to correlations between subsequent avalanches appearing during the motion of one particle.

IV. PROPAGATION OF EULERIAN WALKERS

Besides the evolution of the system as a whole, we can describe the motion of a particle itself. Consider the particle dropped on the lattice with a spanning tree. We call the site i a predecessor of j if the arrow path comes from i to j. Since the particle trajectory is traced by arrows, all visited sites are predecessors of the site of a current particle location. If the particle arrives at the site which is its predecessor, the loop is closed. Thus the particle can visit the sites that have already been visited only during an avalanche.

We divide the motion of the particle into the following stages. The first stage coincides with the first avalanche. At the moment it finishes, the avalanche area remains bounded by the anticlockwise loop opened at the bond connecting two sites where the avalanche begins and where it ends. Further, moving on the lattice, the particle cannot enter the area of the first avalanche until it creates a loop enclosing this area. For this time, new avalanches appear beyond the first one, being attached to its boundary and tending to go clockwise around it. Eventually, the particle creates a loop enclosing the area of the first avalanche and visits it again. When the avalanche corresponding to this loop ends, the second stage of the evolution finishes. At this moment, we have the cluster of visited

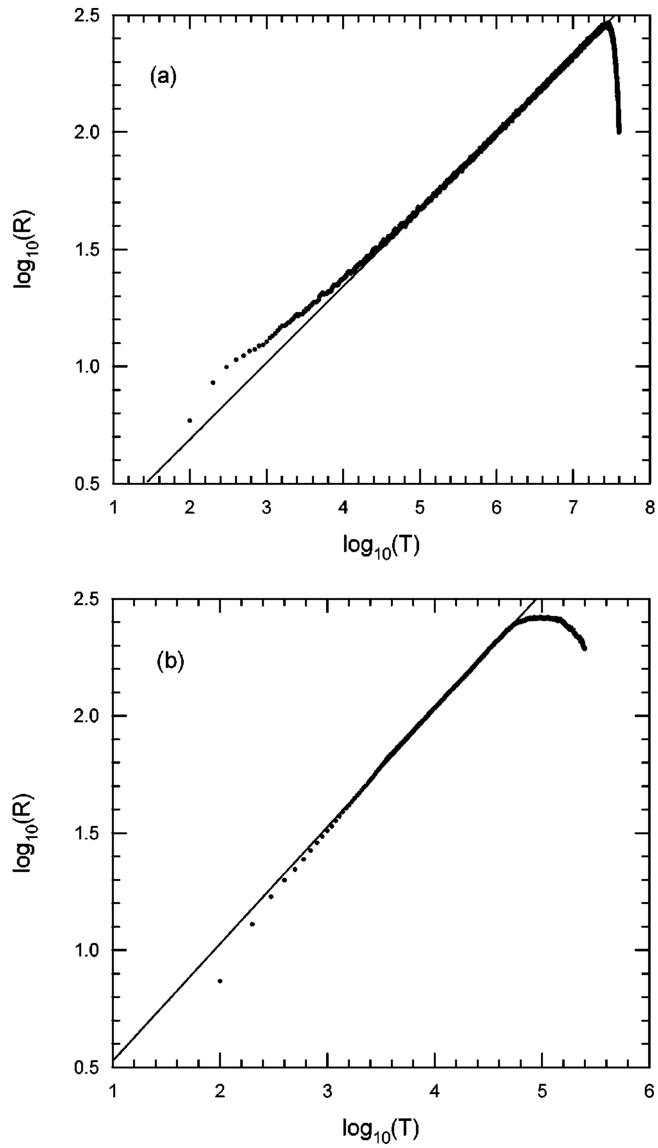

FIG. 4. The dependence of the mean square displacement of the particle on time in the transient (a) and SOC (b) states. The obtained values of the critical exponents are $\nu_t = 0.33$ and $\nu = 0.5$, respectively.

sites which consists of the area of the first avalanche, where each inner site is visited eight times, surrounded by the clusters of subsequent avalanches, where all sites are visited four times (Fig. 3).

Further behavior of the system is similar. If at some evolution stage we have a cluster of visited sites, at the next stage all sites of this cluster will be visited four more times and some new area will be added to the cluster of visited sites. After each evolution stage is completed, the cluster of visited sites is compact because it consists of compactly situated avalanche clusters.

Thus we obtain the system of compact clusters where the sites are visited $4N$, $N = 1,2,\ldots$ times. The clusters are strictly embedded one into another with a growing number of visits like Grassberger-Manna clusters in ASM [15].

Using this picture, we can find the time dependence of the mean square displacement of the particle in the critical state. The number of visits $N(R)$ of a site separated from the origin by the distance R is given by the Green function of the Laplace equation (9). When $|\mathbf{r}-\mathbf{r}'|$ tends to the lattice size L, $G(\mathbf{r},\mathbf{r}')$ decays as $\ln(L/|\mathbf{r}-\mathbf{r}'|)$, so we can write

$$\frac{dN(R)}{dR}\sim-\frac{1}{R}. \tag{15}$$

On the other hand, the time T required for a particle to visit four times all the sites of the compact cluster is of an order of its size R^2. Then, the rate of the growth is

$$\frac{dN}{dT}\sim-\frac{1}{R^2}. \tag{16}$$

Using Eqs. (15) and (16) and the property of compactness of the embedded clusters, we obtain the mean square displacement

$$\langle R^2\rangle\sim T^{2\nu}, \quad \nu=\frac{1}{2} \tag{17}$$

that is the diffusion law of a simple random walk.

In the transient state, we have no spanning tree representation for the evolution of the system. The sites already visited by the particle are connected with the current particle location by an arrow path and the cluster of these sites has an acyclic structure. However, the cluster of acyclic arrows is embedded into the media of randomly distributed arrows.

The particle moves around the cluster of visited sites clockwise, closing the loops and then covering them. Each time, going around the cluster, it visits the sites of cluster four times as in a recurrent state. However, in the transient case, the linear size of increasing the cluster of visited sites does not depend on the size of the cluster, as the arrows beyond it are not correlated. The time of increasing is of an order of the size of the cluster, i.e.,

$$\frac{dR}{dT}\sim\frac{1}{R^2}. \tag{18}$$

Thus, instead of the simple diffusion law (17) in the critical state, one obtains for the transient states [9]

$$\langle R^2\rangle\sim T^{2\nu_t}, \quad \nu_t=\frac{1}{3}. \tag{19}$$

Note that the power law (19) is valid only on the time scale much greater than the time being spent inside the cluster of the visited sites. Inside the cluster, motion of the particles is similar to that in the critical state with the diffusion law (17).

Now we can estimate the average time required to reach the critical state starting from an arbitrary random configuration of arrows. To get a spanning tree on the lattice, the particle must visit all sites at least once. Using Eq. (19) we can obtain for the lattice of the size $L\times L$

$$T_c\sim L^3. \tag{20}$$

The same time is required for a particle walking on the closed graph to settle into the Eulerian trail.

We also measured the mean square displacement numerically. Starting from the transient state, $\langle R^2\rangle$ is described by the power law with the critical exponent $\nu_t=0.33$ as is shown in Fig. 4(a). Subsequent evolution of the system by repeated additions of particles changes this power law. For the system in the SOC state, we obtained the value $\nu=0.5$ [Fig. 4(b)]. These simulations illustrate the exact results obtained above.

In summary, we considered the dynamics of the Eulerian walkers model. The structure of avalanches in the SOC state was studied in detail. We obtained the critical exponent for the distribution of durations of the first avalanche. Considering the evolution of the system as a sequence of avalanches, we found the simple diffusion law for the mean square displacement of the particle in the SOC state. The crossover from the transient state into the SOC state was described qualitatively. The obtained exact results were confirmed by numerical simulations.

ACKNOWLEDGMENTS

We would like to thank Professor D. Dhar for fruitful discussions. We are grateful to E.V. Ivashkevich for useful remarks about the manuscript. Partial financial support by the Russian Foundation for Basic Research under Grant No. 97-01-01030, and by the INTAS under Grant No. 96-690 is acknowledged. One of us (A.M.P.) was partially supported by INTAS under Grant No. 96-0457 within the research program of the International Center for Fundamental Physics in Moscow.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).
[2] V. Frete, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang, and P. Meakin, Nature (London) **379**, 49 (1996).
[3] B. Drossel and F. Schwabl, Phys. Rev. Lett. **69**, 1629 (1992).
[4] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[5] S. N. Majumdar and D. Dhar, J. Phys. A **24**, L357 (1991).
[6] V. B. Priezzhev, J. Stat. Phys. **74**, 955 (1994).
[7] E. V. Ivashkevich, D. V. Ktitarev, and V. B. Priezzhev, Physica A **209**, 347 (1994).
[8] V. B. Priezzhev, D. V. Ktitarev, and E. V. Ivashkevich, Phys. Rev. Lett. **76**, 2093 (1996).
[9] V. B. Priezzhev, D. Dhar, A. Dhar, and S. Krishnamurthy, Phys. Rev. Lett. **77**, 5079 (1996).
[10] F. Harary and E. M. Palmer, *Graphical Enumeration* (Academic Press, New York, 1973).
[11] V. B. Priezzhev, e-print cond-mat/9605094.
[12] R. R. Shcherbakov, Vl. V. Papoyan, and A. M. Povolotsky, Phys. Rev. E **55**, 3686 (1997).
[13] S. S. Manna, D. Dhar, and S. N. Majumdar, Phys. Rev. A **46**, R4471 (1992).
[14] D. Dhar and A. Dhar, e-print cond-mat/9704026.
[15] P. Grassberger and S. S. Manna, J. Phys. (France) **51**, 1077 (1990).